

# Nordhaus-Gaddum-type results for the generalized edge-connectivity of graphs\*

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## Abstract

Let  $G$  be a graph,  $S$  be a set of vertices of  $G$ , and  $\lambda(S)$  be the maximum number  $\ell$  of pairwise edge-disjoint trees  $T_1, T_2, \dots, T_\ell$  in  $G$  such that  $S \subseteq V(T_i)$  for every  $1 \leq i \leq \ell$ . The generalized  $k$ -edge-connectivity  $\lambda_k(G)$  of  $G$  is defined as  $\lambda_k(G) = \min\{\lambda(S) | S \subseteq V(G) \text{ and } |S| = k\}$ . Thus  $\lambda_2(G) = \lambda(G)$ . In this paper, we consider the Nordhaus-Gaddum-type results for the parameter  $\lambda_k(G)$ . We determine sharp upper and lower bounds of  $\lambda_k(G) + \lambda_k(\overline{G})$  and  $\lambda_k(G) \cdot \lambda_k(\overline{G})$  for a graph  $G$  of order  $n$ , as well as for a graph of order  $n$  and size  $m$ . Some graph classes attaining these bounds are also given.

**Keywords:** edge-connectivity; Steiner tree; edge-disjoint trees; generalized edge-connectivity; complementary graph.

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## 1 Introduction

All graphs considered in this paper are undirected, finite and simple. We refer to the book [4] for graph theoretical notation and terminology not described here. For a graph  $G(V, E)$  and a set  $S \subseteq V$  of at least two vertices, an  $S$ -Steiner tree or an Steiner tree connecting  $S$  (Shortly, a Steiner tree) is a subgraph  $T(V', E')$  of  $G$  which is a tree such that  $S \subseteq V'$ . Two Steiner trees  $T$  and  $T'$  connecting  $S$  are *edge-disjoint* if  $E(T) \cap E(T') = \emptyset$ . The *Steiner Tree Packing Problem* for a given graph  $G(V, E)$  and  $S \subseteq V(G)$  asks to find a set of maximum number of edge-disjoint  $S$ -Steiner trees in  $G$ . This problem has obtained wide attention and many results have been worked out, see [18, 19, 20]. The problem for  $S = V(G)$  is called the *Spanning Tree Packing Problem*. For any graph  $G$  of order  $n$ , the *spanning tree packing number* or *STP number*, is the maximum number of edge-disjoint spanning trees contained in  $G$ . For the *STP* number, Palmer gave a good survey, see [17].

Recently, we introduced the concept of generalized edge-connectivity of a graph  $G$  in [13]. For  $S \subseteq V(G)$ , the *generalized local edge-connectivity*  $\lambda(S)$  is the maximum number of edge-disjoint trees in  $G$  connecting  $S$ . Then the *generalized  $k$ -edge-connectivity*  $\lambda_k(G)$

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of  $G$  is defined as  $\lambda_k(G) = \min\{\lambda(S) : S \subseteq V(G) \text{ and } |S| = k\}$ . Thus  $\lambda_2(G) = \lambda(G)$ . Set  $\lambda_k(G) = 0$  when  $G$  is disconnected. We call it the generalized  $k$ -edge-connectivity since Chartrand et al. in [5] introduced the concept of generalized (vertex) connectivity in 1984. There have been many results on the generalized connectivity, see [10, 11, 12, 13].

One can see that the Steiner Tree Packing Problem studies local properties of graphs, but the generalized edge-connectivity focuses on global properties of graphs. Actually, the *STP* number of a graph  $G$  is just  $\lambda_n(G)$ .

In addition to being natural combinatorial measures, the Steiner Tree Packing Problem and the generalized edge-connectivity can be motivated by their interesting interpretation in practice as well as theoretical consideration. For the practical backgrounds, we refer to [7, 8, 15].

From a theoretical perspective, both extremes of this problem are fundamental theorems in combinatorics. One extreme of the problem is when we have two terminals. In this case internally (edge-)disjoint trees are just internally (edge-)disjoint paths between the two terminals, and so the problem becomes the well-known Menger theorem. The other extreme is when all the vertices are terminals. In this case internally disjoint trees and edge-disjoint trees are just spanning trees of the graph, and so the problem becomes the classical Nash-Williams-Tutte theorem.

**Theorem 1.** (Nash-Williams [14], Tutte [16]) *A multigraph  $G$  contains a system of  $\ell$  edge-disjoint spanning trees if and only if*

$$\|G/\mathcal{P}\| \geq \ell(|\mathcal{P}| - 1)$$

*holds for every partition  $\mathcal{P}$  of  $V(G)$ , where  $\|G/\mathcal{P}\|$  denotes the number of crossing edges in  $G$ , i.e., edges between distinct parts of  $\mathcal{P}$ .*

**Corollary 1.** *Every  $2\ell$ -edge-connected graph contains a system of  $\ell$  edge-disjoint spanning trees.*

Let  $\mathcal{G}(n)$  denote the class of simple graphs of order  $n$  and  $\mathcal{G}(n, m)$  the subclass of  $\mathcal{G}(n)$  having  $m$  edges. Give a graph theoretic parameter  $f(G)$  and a positive integer  $n$ , the Nordhaus-Gaddum (**N-G**) Problem is to determine sharp bounds for: (1)  $f(G) + f(\overline{G})$  and (2)  $f(G) \cdot f(\overline{G})$ , as  $G$  ranges over the class  $\mathcal{G}(n)$ , and characterize the extremal graphs. The Nordhaus-Gaddum type relations have received wide investigations. Recently, Aouchiche and Hansen published a survey paper on this subject, see [3].

In this paper, we study  $\lambda_k(G) + \lambda_k(\overline{G})$  and  $\lambda_k(G) \cdot \lambda_k(\overline{G})$  for the parameter  $\lambda_k(G)$  where  $G \in \mathcal{G}(n)$  and  $G \in \mathcal{G}(n, m)$ .

## 2 Nordhaus-Gaddum-type results in $\mathcal{G}(n)$

The following observation is easily seen.

**Observation 1.** (1) *If  $G$  is a connected graph, then  $1 \leq \lambda_k(G) \leq \lambda(G) \leq \delta(G)$ ;*  
(2) *If  $H$  is a spanning subgraph of  $G$ , then  $\lambda_k(H) \leq \lambda_k(G)$ .*

(3) Let  $G$  be a connected graph with minimum degree  $\delta$ . If  $G$  has two adjacent vertices of degree  $\delta$ , then  $\lambda_k(G) \leq \delta - 1$ .

Alavi and Mitchem in [2] considered Nordhaus-Gaddum-type results for the connectivity and edge-connectivity parameters. In [13] we were concerned with analogous inequalities involving the generalized  $k$ -connectivity and generalized  $k$ -edge-connectivity. We showed that  $1 \leq \lambda_k(G) + \lambda_k(\overline{G}) \leq n - \lceil k/2 \rceil$ , but this is just a starting result and now we will further study the Nordhaus-Gaddum type relations.

To start with, let us recall the Harary graph  $H_{n,d}$  on  $n$  vertices, which is constructed by arranging the  $n$  vertices in circular order and spreading the  $d$  edges around the boundary in a nice way, keeping the chords as short as possible. They have the maximum connectivity for their size and  $\kappa(H_{n,d}) = \lambda(H_{n,d}) = \delta(H_{n,d}) = d$ . Palmer [17] gave the *STP* number of some special graph classes.

**Lemma 1.** [17] (1) The *STP* number of a complete bipartite graph  $K_{a,b}$  is  $\lfloor \frac{ab}{a+b-1} \rfloor$ .

(2) The *STP* number of a Harary graph  $H_{n,d}$  is  $\lfloor d/2 \rfloor$ .

Corresponding to (1) of Observation 1, we can obtain a sharp lower bound for the generalized  $k$ -edge-connectivity by Corollary 1. Actually, a connected graph  $G$  contains  $\lfloor \frac{1}{2}\lambda(G) \rfloor$  spanning trees. Each of them is also a Steiner tree connecting  $S$ . So the following proposition is immediate.

**Proposition 1.** For a connected graph  $G$  of order  $n$  and  $3 \leq k \leq n$ ,  $\lambda_k(G) \geq \lfloor \frac{1}{2}\lambda(G) \rfloor$ . Moreover, the lower bound is sharp.

In order to show the sharpness of this lower bound for  $k = n$ , we consider the Harary graph  $H_{n,2r}$ . Clearly,  $\lambda(G) = 2r$ . From (2) of Lemma 1,  $H_{n,2r}$  contains  $r$  spanning trees, that is,  $\lambda_n(H_{n,2r}) = r$ . So  $\lambda_n(H_{n,2r}) = \lfloor \frac{1}{2}\lambda(G) \rfloor$ . For general  $k$  ( $3 \leq k \leq n$ ), one can check that the cycle  $C_n$  can attain the lower bound since  $\frac{1}{2}\lambda(C_n) = 1 = \lambda_k(C_n)$ .

The following proposition indicates that the monotone properties of  $\lambda_k$ , that is,  $\lambda_n \leq \lambda_{n-1} \leq \dots \leq \lambda_4 \leq \lambda_3 \leq \lambda$ , is true for  $2 \leq k \leq n$ .

**Proposition 2.** For two integers  $k$  and  $n$  with  $2 \leq k \leq n - 1$ , and a connected graph  $G$ ,  $\lambda_{k+1}(G) \leq \lambda_k(G)$ .

*Proof.* Assume  $3 \leq k \leq n - 1$ . Set  $\lambda_{k+1}(G) = \ell$ . For each  $S \subseteq V(G)$  with  $|S| = k$ , we let  $S' = S \cup \{u\}$ , where  $u \notin S$ . Since  $\lambda_{k+1}(G) = \ell$ , there exist  $\ell$  edge-disjoint trees connecting  $S'$ . These trees are also  $\ell$  edge-disjoint trees connecting  $S$ . So  $\lambda_k(G) \geq \ell$  and  $\lambda_{k+1}(G) \leq \lambda_k(G)$ . Combining this with (1) of Observation 1, we get that  $\lambda_{k+1}(G) \leq \lambda_k(G)$  for  $2 \leq k \leq n - 1$ .  $\square$

Now we give the lower bounds of  $\lambda_k(G) + \lambda_k(\overline{G})$  and  $\lambda_k(G) \cdot \lambda_k(\overline{G})$ .

**Lemma 2.** Let  $G \in \mathcal{G}(n)$ . Then

(1)  $\lambda_k(G) + \lambda_k(\overline{G}) \geq 1$ ;

(2)  $\lambda_k(G) \cdot \lambda_k(\overline{G}) \geq 0$ .

Moreover, the two lower bounds are sharp.

*Proof.* (1) If  $\lambda_k(G) + \lambda_k(\overline{G}) = 0$ , then  $\lambda_k(G) = \lambda_k(\overline{G}) = 0$ , that is,  $G$  and  $\overline{G}$  are all disconnected, which is impossible, and so  $\lambda_k(G) + \lambda_k(\overline{G}) \geq 1$ .

(2) By definition,  $\lambda_k(G) \geq 0$  and  $\lambda_k(\overline{G}) \geq 0$ , and so  $\lambda_k(G) \cdot \lambda_k(\overline{G}) \geq 0$ .  $\square$

The following observation indicates the graphs attaining the lower bound of (1) in Lemma 2.

**Observation 2.**  $\lambda_k(G) \cdot \lambda_k(\overline{G}) = 0$  if and only if  $G$  or  $\overline{G}$  is disconnected.

In [13] we obtained the exact value of the generalized  $k$ -edge-connectivity of a complete graph  $K_n$ .

**Lemma 3.** [13] For two integers  $n$  and  $k$  with  $2 \leq k \leq n$ ,  $\lambda_k(K_n) = n - \lceil k/2 \rceil$ .

For a connected graph  $G$  of order  $n$ , we know that  $1 \leq \lambda_k(G) \leq \lambda_k(K_n) = n - \lceil k/2 \rceil$ . In [13] we characterized the graphs attaining the upper bound.

**Lemma 4.** [13] For a connected graph  $G$  of order  $n$  with  $3 \leq k \leq n$ ,  $\lambda_k(G) = n - \lceil \frac{k}{2} \rceil$  if and only if  $G = K_n$  for  $k$  even;  $G = K_n \setminus M$  for  $k$  odd, where  $M$  is an edge set such that  $0 \leq |M| \leq \frac{k-1}{2}$ .

As we know, it is difficult to characterize the graphs with  $\lambda_k(G) = 1$ , even with  $\lambda_3(G) = 1$ . So we want to add some conditions to attack such a problem. Motivated by such an idea, we hope to characterize the graphs with  $\lambda_k(G) + \lambda_k(\overline{G}) = 1$ . Actually, the Norhaus-Gaddum-type problems also need to characterize the extremal graphs attaining the bounds.

Before studying the lower bounds of  $\lambda_k(G) + \lambda_k(\overline{G})$  and  $\lambda_k(G) \cdot \lambda_k(\overline{G})$ , we give some graph classes (Every element of each graph class has order  $n$ ), which will be used later.

For  $n \geq 5$ ,  $\mathcal{G}_n^1$  is a graph class as shown in Figure 1 (a) such that  $\lambda(G) = 1$  and  $d_G(v_1) = n - 1$  for  $G \in \mathcal{G}_n^1$ , where  $v_1 \in V(G)$ ;  $\mathcal{G}_n^2$  is a graph class as shown in Figure 1 (b) such that  $\lambda(G) = 2$  and  $d_G(u_1) = n - 1$  for  $G \in \mathcal{G}_n^2$ , where  $u_1 \in V(G)$ ;  $\mathcal{G}_n^3$  is a graph class as shown in Figure 1 (c) such that  $\lambda(G) = 2$  and  $d_G(v_1) = n - 1$  for  $G \in \mathcal{G}_n^3$ , where  $v_1 \in V(G)$ ;  $\mathcal{G}_n^4$  is a graph class as shown in Figure 1 (d) such that  $\lambda(G) = 2$ .

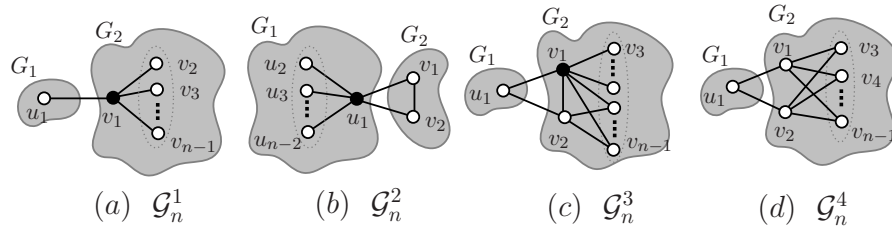


Figure 1. Graphs for Proposition 3 (The degree of a black vertex is  $n - 1$ ).

The following observation and lemma are some preparations for Proposition 3.

For  $n \geq 5$ , let  $K_{2,n-2}^+$  and  $K_{2,n-2}^{++}$  be two graphs obtained from the complete bipartite graph  $K_{2,n-2}$  by adding one and two edges on the part having  $n - 2$  vertices, respectively.

**Observation 3.** (1)  $\lambda_n(K_{2,n-2}^{++}) \geq 2$ ; (2)  $\lambda_{n-1}(K_{2,n-2}^+) \geq 2$ ,  $\lambda_n(K_{2,n-2}^+) = 1$ ; (3)  $\lambda_{n-2}(K_{2,n-2}) \geq 2$ ,  $\lambda_n(K_{2,n-2}) = \lambda_{n-1}(K_{2,n-2}) = 1$ .

*Proof.* (1) As shown in Figure 2 (a),  $\lambda_n(K_{2,n-2}^{++}) \geq 2$ .

(2) As shown in Figure 2 (b), we have  $\lambda_{n-1}(K_{2,n-2}^+) \geq 2$ . Since  $|E(K_{2,n-2}^+)| = 2(n-2) + 1$ ,  $\lambda_n(K_{2,n-2}^+) \leq \lfloor \frac{2(n-2)+1}{n-1} \rfloor$ , which implies that  $\lambda_n(K_{2,n-2}^+) \leq 1$ . Since  $K_{2,n-2}^+$  is connected,  $\lambda_n(K_{2,n-2}^+) = 1$ .

(3) As shown in Figure 2 (c), it follows that  $\lambda_{n-2}(K_{2,n-2}) \geq 2$ . Let  $U = \{u_1, u_2\}$  and  $W = \{w_1, w_2, \dots, w_{n-2}\}$  be two parts of the complete bipartite graph  $K_{2,n-2}$ . Choose  $S = \{u_1, u_2, w_1, w_2, \dots, w_{n-3}\}$ . If there exists an  $S$ -tree containing vertex  $w_{n-2}$ , then this tree will use  $n-1$  edges of  $E(K_{2,n-2})$ , which implies that  $\lambda_{n-1}(K_{2,n-2}) \leq 1$  since  $|E(K_{2,n-2})| = 2(n-2)$ . Suppose that there is no  $S$ -tree containing vertex  $w_2$ . Pick up a such tree, say  $T$ . Then there exists a vertex of degree 2 in  $T$ , which implies that there is no other  $S$ -tree in  $K_{2,n-2}$ . So  $\lambda_{n-1}(K_{2,n-2}) \leq 1$ . Since  $K_{2,n-2}$  is connected,  $\lambda_{n-1}(K_{2,n-2}) = 1$ . From Proposition 2,  $\lambda_n(K_{2,n-2}) = 1$ .  $\square$

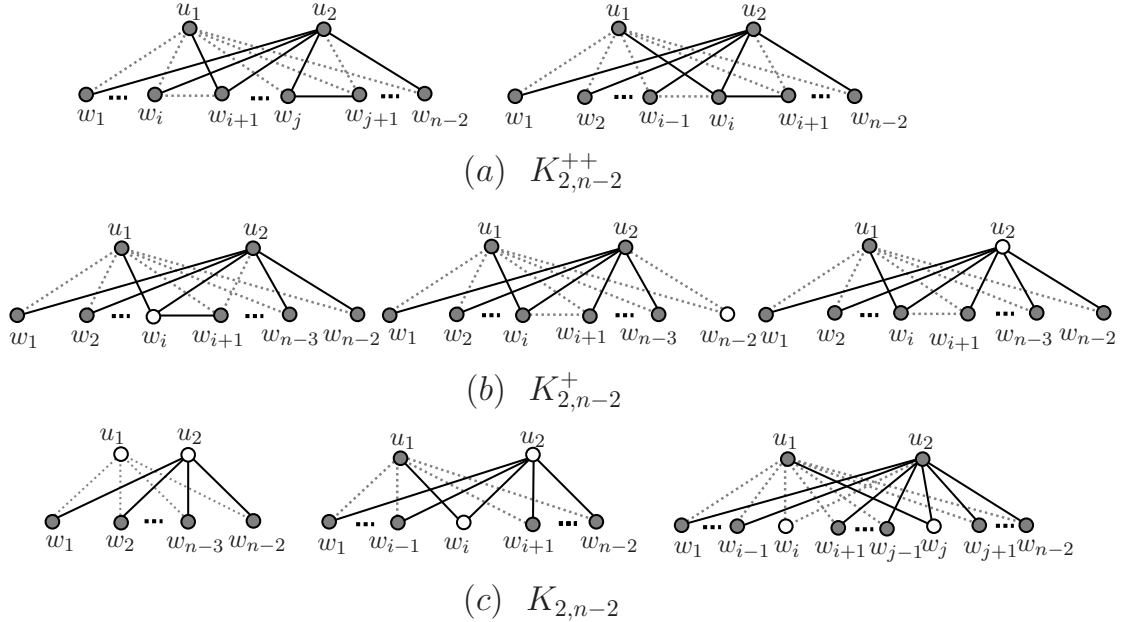


Figure 2. Graphs for Observation 2.

**Lemma 5.** Let  $G$  be a connected graph. If  $\lambda(G) = 3$  and there exists a vertex  $u \in V(G)$  such that  $d_G(u) = n-1$ , then  $\lambda_k(G) \geq 2$  for  $3 \leq k \leq n$ .

*Proof.* Let  $G_1, \dots, G_r$  be the connected components of  $G \setminus u$ . Since  $\lambda(G) = 3$ , it follows that  $\delta(G_i) \geq 2$  ( $1 \leq i \leq r$ ). Let  $|V(G_i)| = n_i$  ( $1 \leq i \leq r$ ) and  $V(G_i) = \{v_{i1}, v_{i2}, \dots, v_{in_i}\}$ . Then there exists an edge, without loss of generality, say  $e_i = v_{i1}v_{i2} \in E(G_i)$  such that  $G_i \setminus e_i$  is connected for  $1 \leq i \leq r$ . Thus  $G_i \setminus e_i$  contains a spanning tree, say  $T_i$  ( $1 \leq i \leq r$ ). The trees  $T = uv_{11} \cup T_1 \cup uv_{21} \cup T_2 \cup \dots \cup uv_{r1} \cup T_r$  and  $T' = v_{11}v_{12} \cup uv_{12} \cup \dots \cup uv_{1n_1} \cup v_{21}v_{22} \cup uv_{22} \cup \dots \cup uv_{2n_2} \cup \dots \cup v_{r1}v_{r2} \cup uv_{r2} \cup \dots \cup uv_{rn_r}$  are two spanning trees of  $G$ , that is,  $\lambda_n(G) \geq 2$ . Combining this with Proposition 2,  $\lambda_k(G) \geq 2$  for  $3 \leq k \leq n$ .  $\square$

**Proposition 3.**  $\lambda_k(G) + \lambda_k(\overline{G}) = 1$  if and only if  $G$  (symmetrically,  $\overline{G}$ ) satisfies one of the following conditions:

- (1)  $G \in \mathcal{G}_n^1$  or  $G \in \mathcal{G}_n^2$ ;
- (2)  $G \in \mathcal{G}_n^3$  and there exists a component  $G_i$  of  $G \setminus v_1$  such that  $G_i$  is a tree and  $|V(G_i)| < k$ ;
- (3)  $G \in \{K_{2,n-2}^+, K_{2,n-2}\}$  for  $k = n$  and  $n \geq 5$ , or  $G \in \{P_3, C_3\}$  for  $k = n = 3$ , or  $G \in \{C_4, K_4 \setminus e\}$  for  $k = n = 4$ , or  $G = K_{3,3}$  for  $k = n = 6$ , or  $G = K_{2,n-2}$  for  $k = n - 1$  and  $n \geq 5$ , or  $G = C_4$  for  $k = n - 1 = 3$ .

*Proof. Necessity.* Let  $G$  be a graph satisfying one of the conditions of (1), (2) and (3). One can see that  $G$  is connected and its complement  $\overline{G}$  is disconnected. Thus  $\lambda_k(G) + \lambda_k(\overline{G}) = \lambda_k(G)$  and  $\lambda_k(G) \geq 1$ . We only need to show that  $\lambda_k(G) \leq 1$  for each graph  $G$  satisfying one of the conditions of (1), (2) and (3). For  $G \in \mathcal{G}_n^1$ , since  $\delta(G) = 1$  we have  $\lambda_k(G) \leq 1$  by (1) of Observation 1. For  $G \in \mathcal{G}_n^2$ , it follows that  $\lambda_k(G) \leq \delta(G) - 1 = 1$  by (3) of Observation 1 since  $d_G(v_1) = d_G(v_2) = \delta(G) = 2$ . Suppose  $G \in \mathcal{G}_n^3$  and there exists a connected component  $G_i$  of  $G \setminus v_1$  such that  $G_i$  is a tree and  $|V(G_i)| < k$ . Set  $V(G_i) = \{v_{i1}, v_{i2}, \dots, v_{in_i}\}$ . We choose  $S \subseteq V(G)$  such that  $V(G_i) \cup \{v_1\} = S' \subseteq S$ . Then  $|E(G[S'])| = 2n_i - 1$ . Since every spanning tree of  $G[S']$  uses  $n_i - 1$  edges of  $E(G[S'])$ , there exists at most one spanning tree of  $G[S']$ , which implies that there is at most one tree connecting  $S$  in  $G$ . So  $\lambda_k(G) \leq 1$ . For  $G = K_{2,n-2}^+$ ,  $\lambda_n(G) = 1$  by (2) of Observation 3. For  $G = K_{2,n-2}$ , by (3) of Observation 3, we have  $\lambda_n(K_{2,n-2}) = \lambda_{n-1}(K_{2,n-2}) = 1$ . For  $G = K_{3,3}$ ,  $\lambda_n(G) \leq \lfloor \frac{|E(G)|}{n-1} \rfloor = \lfloor \frac{9}{5} \rfloor = 1$ . For  $G \in \{P_3, C_3, C_4, K_4 \setminus e\}$ , one can check that  $\lambda_k(G) \leq 1$  for  $k = n$  or  $k = n - 1$ . From these together with  $\lambda_k(G) \geq 1$ , we have  $\lambda_k(G) + \lambda_k(\overline{G}) = \lambda_k(G) = 1$ .

*Sufficiency.* Suppose  $\lambda_k(G) + \lambda_k(\overline{G}) = 1$ . Then  $\lambda_k(G) = 1$  and  $\lambda_k(\overline{G}) = 0$ , or  $\lambda_k(\overline{G}) = 1$  and  $\lambda_k(G) = 0$ . By symmetry, without loss of generality, we let  $\lambda_k(G) = 1$  and  $\lambda_k(\overline{G}) = 0$ . From these together with Proposition 1,  $\lambda(\overline{G}) = 0$  and  $1 \leq \lambda(G) \leq 3$ . So we have the following three cases to consider.

*Case 1.*  $\lambda(G) = 1$ .

For  $n = 3$ , one can check that  $G = P_3$  satisfies  $\lambda(G) = 1$  but  $\lambda(\overline{G}) = 0$ . Now we assume  $n \geq 4$ . Since  $\lambda(G) = 1$ , there exists at least one cut edge in  $G$ , say  $e = u_1v_1$ . Let  $G_1$  and  $G_2$  be two connected components of  $G \setminus e$  such that  $u_1 \in V(G_1)$  and  $v_1 \in V(G_2)$ . Set  $V(G_1) = \{u_1, u_2, \dots, u_{n_1}\}$  and  $V(G_2) = \{v_1, v_2, \dots, v_{n_2}\}$ , where  $n_1 + n_2 = n$ . Suppose  $n_i \geq 2$  ( $i = 1, 2$ ). For any  $u_i, u_j \in V(G_1)$ ,  $u_i$  and  $u_j$  are connected in  $\overline{G}$  since there exists a path  $u_i v_2 u_j$  in  $\overline{G}$ ; for any  $v_i, v_j \in V(G_2)$ ,  $v_i$  and  $v_j$  are connected in  $\overline{G}$  since there exists a path  $v_i u_2 v_j$  in  $\overline{G}$ ; for any  $u_i \in V(G_1)$  and  $v_j \in V(G_2)$  ( $i \neq 1$  or  $j \neq 1$ ),  $u_i v_j \in E(\overline{G})$ . Clearly, the path  $u_1 v_2 u_2 v_1$  connects  $u_1$  and  $v_1$  in  $\overline{G}$ . So  $\overline{G}$  is connected, a contradiction. Thus  $n_1 = 1$  or  $n_2 = 1$ . Without loss of generality, let  $n_1 = 1$ . Then  $V(G_1) = \{u_1\}$  and  $V(G_2) = \{v_1, v_2, \dots, v_{n-1}\}$ . Clearly,  $G$  is a graph obtained from  $G_2$  by attaching the edge  $e = u_1v_1$ . Since  $u_1v_j \notin E(G)$  ( $1 \leq j \leq n-1$ ),  $u_1v_j \in E(\overline{G})$ . If  $d_G(v_1) \leq n-2$ , then there exists one vertex  $v_j$  such that  $v_1v_j \in E(\overline{G})$ , which results in  $\lambda(\overline{G}) \geq 1$ , a contradiction. So  $d_G(v_1) = n-1$  and  $G \in \mathcal{G}_n^1$  (See Figure 1 (a)).

*Case 2.*  $\lambda(G) = 2$ .



For  $n = 3, 4$ , the graph  $G \in \{C_3, C_4, K_4 \setminus e\}$  satisfies that  $\lambda(G) = 2$  and  $\lambda(\overline{G}) = 0$ . Since  $\lambda_3(C_3) = 1$ ,  $\lambda_3(C_4) = 1$ ,  $\lambda_4(C_4) = 1$ ,  $\lambda_3(K_4 \setminus e) = 2$  and  $\lambda_4(K_4 \setminus e) = 1$ , we have  $G = C_3$  for  $k = n = 3$ ;  $G \in \{C_4, K_4 \setminus e\}$  for  $k = n = 4$ ;  $G = C_4$  for  $k = n - 1 = 3$ . Now we assume  $n \geq 5$ . Since  $\lambda(G) = 2$ , there exists an edge cut  $M$  such that  $|M| = 2$ . Let  $G_1$  and  $G_2$  be two connected components of  $G \setminus M$ ,  $V(G_1) = \{u_1, \dots, u_{n_1}\}$  and  $V(G_2) = \{v_1, \dots, v_{n_2}\}$ , where  $n_1 + n_2 = n$ . Clearly,  $G[M] = 2K_2$  or  $G[M] = P_3$ .

At first, we consider the case  $G[M] = 2K_2$ . Without loss of generality, let  $M = \{u_1v_1, u_2v_2\}$ . Since  $n \geq 5$ ,  $n_1 \geq 3$  or  $n_2 \geq 3$ . Without loss of generality, let  $n_1 \geq 3$ . Clearly, any two vertices  $v_i, v_j \in V(G_2)$  are connected in  $\overline{G}$  since there exists a path  $v_iu_3v_j$  in  $\overline{G}$ . Furthermore, for any  $u_i \in V(G_1)$ ,  $u_iv_1 \in E(\overline{G})$  or  $u_iv_2 \in E(\overline{G})$ . So  $\overline{G}$  is connected and  $\lambda(\overline{G}) \geq 1$ , a contradiction.

Next, we consider the case  $G[M] = P_3$ . Without loss of generality, let  $P = v_1u_1v_2$  be the path of order 3. Since  $n \geq 5$ , there exist at least two vertices in  $G \setminus \{u_1, v_1, v_2\}$ . If  $n_1 \geq 2$  and  $n_2 \geq 3$ , then we can check that  $\overline{G}$  is connected, a contradiction. So we assume that  $n_1 = 1$  or  $n_2 = 2$ , that is,  $V(G_2) = \{v_1, v_2\}$  or  $V(G_1) = \{u_1\}$ .

For the former,  $V(G_1) = \{u_1, u_2, \dots, u_{n-2}\}$ . Since  $\lambda(G) = 2$ ,  $v_1v_2 \in E(G)$ . Clearly,  $v_1u_j, v_2u_j \notin E(G)$  ( $2 \leq j \leq n-2$ ), which implies that  $v_1u_j, v_2u_j \in E(\overline{G})$ . Therefore,  $u_1u_j \notin E(\overline{G})$  ( $2 \leq j \leq n-2$ ) since  $\overline{G}$  is disconnected. Thus  $u_1u_j \in E(G)$  for each  $j$  ( $2 \leq j \leq n-2$ ). So  $d_G(u_1) = n-1$  and  $G \in \mathcal{G}_n^2$  (See Figure 1 (b)).

For the latter, let  $V(G_2) = \{v_1, v_2, \dots, v_{n-1}\}$ . First we consider the case  $v_1v_2 \in E(G)$ . Since  $u_1v_j \notin E(G)$  ( $3 \leq j \leq n-1$ ), we have  $u_1v_j \in E(\overline{G})$ . If  $3 \leq d_G(v_1) \leq n-2$  and  $3 \leq d_G(v_2) \leq n-2$ , then there exist two vertices  $v_i$  and  $v_j$  such that  $v_1v_i, v_2v_j \in E(\overline{G})$  ( $3 \leq i, j \leq n-1$ ), which implies that  $\overline{G}$  is connected, a contradiction. So  $d_G(v_1) = n-1$  or  $d_G(v_2) = n-1$ . Without loss of generality, let  $d_G(v_1) = n-1$ . Thus  $G \in \mathcal{G}_n^3$  (See Figure 1 (c)). Now we focus on the graph  $G \setminus v_1$ . Let  $G_1, G_2, \dots, G_r$  be the connected components of  $G \setminus v_1$  and  $V(G_i) = \{v_{i1}, v_{i2}, \dots, v_{in_i}\}$  ( $1 \leq i \leq r$ ), where  $\sum_{i=1}^r n_i = n-1$ . If there exists some connected component  $G_i$  such that  $G_i = K_2$ , then  $G \in \mathcal{G}_n^2$  (See Figure 1 (b)). So we assume  $n_i \geq 3$ . Then we prove the following claim and get a contradiction.

**Claim 1.** For each connected component  $G_i$  of  $G \setminus v_1$ , if  $n_i \geq k$ , or  $n_i \leq k-1$  and  $|E(G_i)| \geq n_i$ , then  $\lambda_k(G) \geq 2$  for  $3 \leq k \leq n$ .

*Proof of Claim 1.* For an arbitrary  $S \subseteq V(G)$  with  $|S| = k$ , we only prove  $\lambda(S) \geq 2$  for  $v_1 \notin S$ . The case for  $v_1 \in S$  can be proved similarly. If there exists some connected component  $G_i$  such that  $S = V(G_i)$ , then  $n_i = k$  and  $G_i$  has a spanning tree, say  $T_i$ . It is also a Steiner tree connecting  $S$ . Since  $T'_i = v_1v_{i1} \cup v_1v_{i2} \dots \cup v_1v_{in_i}$  is another Steiner tree connecting  $S$  and  $T_i, T'_i$  are two edge-disjoint trees, we have  $\lambda(S) \geq 2$ . Let us assume now  $S \neq V(G_i)$  for  $n_i \geq k$  ( $1 \leq i \leq r$ ). Let  $S_i = S \cap V(G_i)$  ( $1 \leq i \leq r$ ) and  $|S_i| = k_i$ . Clearly,  $\bigcup_{i=1}^r S_i = S$  and  $\sum_{i=1}^r k_i = k$ . Thus  $S_i \subset V(G_i)$  for each connected component  $G_i$  such that  $n_i \geq k$ , and  $S_j \subseteq V(G_j)$  for each connected component  $G_j$  such that  $n_j \leq k-1$  and  $|E(G_j)| \geq n_j$ . We will show that there are two edge-disjoint Steiner trees connecting  $S_i \cup \{v_1\}$  in  $G[S_i \cup \{v_1\}]$  for each  $i$  ( $1 \leq i \leq r$ ) so that we can combine these trees to form two edge-disjoint Steiner trees connecting  $S$  in  $G$ . Suppose that  $G_i$  is a connected component such that  $n_i \geq k$ . Note that  $V(G_i) = \{v_{i1}, v_{i2}, \dots, v_{in_i}\}$ . Since  $S_i \subset V(G_i)$ , there exists a vertex, without loss of generality, say  $v_{i1}$ , such that  $v_{i1} \notin S_i$ . Clearly,  $G_i$  contains a

spanning tree, say  $T'_{i1}$ . Thus  $T_{i1} = v_1v_{i1} \cup T'_{i1}$  is a Steiner tree connecting  $S_i \cup \{v_1\}$  in  $G[G_i \cup \{v_1\}]$ . Since  $T_{i2} = v_1v_{i2} \cup v_1v_{i3} \cup \dots \cup v_1v_{in_i}$  is another Steiner tree connecting  $S_i \cup \{v_1\}$ . Clearly,  $T_{i1}$  and  $T_{i2}$  are edge-disjoint. Assume that  $G_j$  is a connected component such that  $n_j \leq k-1$  and  $|E(G_j)| \geq n_j$ . Note that  $V(G_j) = \{v_{j1}, v_{j2}, \dots, v_{jn_j}\}$ . Then there exists an edge, without loss of generality, say  $e_j = v_{j1}v_{j2} \in E(G_j)$  such that  $G_j \setminus e_j$  contains a spanning tree of  $G_j$ , say  $T'_{j1}$ . Thus  $T_{j1} = v_1v_{j1} \cup T'_{j1}$  and  $T_{j2} = v_{j1}v_{j2} \cup v_1v_{j2} \cup \dots \cup v_1v_{jn_j}$  are two edge-disjoint Steiner trees connecting  $S_j \cup \{v_1\}$ . Now we combine these small trees connecting  $S_i \cup \{v_1\}$  ( $1 \leq i \leq r$ ) by the vertex  $v_1$  to form two big trees connecting  $S$ . Clearly,  $T_1 = T_{11} \cup T_{21} \cup \dots \cup T_{r1}$  and  $T_2 = T_{12} \cup T_{22} \cup \dots \cup T_{r2}$  are our desired trees, that is,  $\lambda(S) \geq 2$ . From the arbitrariness of  $S$ , we have  $\lambda_k(G) \geq 2$ .  $\square$

By Claim 1, we know that  $G \in \mathcal{G}_n^3$  and there exists a connected component  $G_i$  of  $G \setminus \{v_1\}$  such that  $n_i \leq k-1$  and  $G_i$  is a tree.

We next consider the case  $v_1v_2 \notin E(G)$  (See Figure 1 (d)). Thus  $v_1v_2 \in E(\overline{G})$ . Since  $u_1v_j \notin E(G)$  ( $3 \leq j \leq n-1$ ),  $u_1v_j \in E(\overline{G})$ , which results in  $v_1v_j, v_2v_j \notin E(\overline{G})$  since  $\overline{G}$  is disconnected. Thus  $v_1v_j, v_2v_j \in E(G)$  for each  $j$  ( $3 \leq j \leq n-1$ ). Let  $R = \{v_j | 3 \leq j \leq n-1\}$ . If  $|E(G[R])| \geq 2$ , then  $G$  contains a subgraph  $K_{2,n-2}^+$ , which implies that  $\lambda_n(G) \geq 2$  by (1) of Observation 3. Combining this with Proposition 2,  $\lambda_k(G) \geq 2$  for  $3 \leq k \leq n$ , a contradiction. If  $|E(G[R])| < 2$ , then  $G = K_{2,n-2}$  and  $K_{2,n-2}^+$ . From Observation 3 and Proposition 2, we have  $\lambda_k(K_{2,n-2}^+) \geq 2$  for  $3 \leq k \leq n-1$  and  $\lambda_k(K_{2,n-2}) \geq 2$  for  $3 \leq k \leq n-2$ , a contradiction. So  $G = K_{2,n-2}^+$  for  $k = n$ , or  $G = K_{2,n-2}$  for  $k = n$ , or  $G = K_{2,n-2}$  for  $k = n-1$ .

*Case 3.*  $\lambda(G) = 3$ .

For  $n = 4$ ,  $G = K_4$ ,  $\lambda_3(G) = \lambda_4(G) = 2$  by Lemma 3. Then  $\lambda_k(G) \geq 2$ , a contradiction. Assume  $n \geq 5$ . Since  $\lambda(G) = 3$ , there exists an edge cut  $M$  such that  $|M| = 3$ . Let  $G_1$

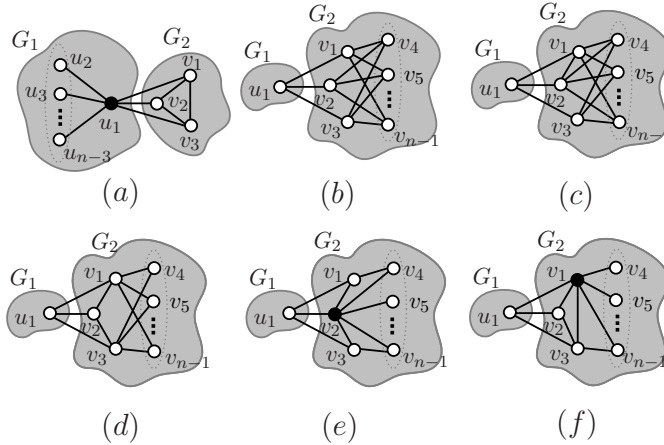


Figure 3. Graphs for Case 3 of Proposition 3.

and  $G_2$  be two connected components of  $G \setminus M$ ,  $V(G_1) = \{u_1, u_2, \dots, u_{n_1}\}$  and  $V(G_2) = \{v_1, v_2, \dots, v_{n_2}\}$ , where  $n_1 + n_2 = n$ . Clearly,  $G[M] = P_4$  or  $G[M] = P_3 \cup K_2$  or  $G[M] = 3K_2$  or  $G[M] = K_{1,n-3}$ . For the former three cases,  $n_i \geq 3$  ( $i = 1, 2$ ) and  $n \geq 6$  since  $\lambda(G) = 3$ . To shorten the discussion, we only prove  $\lambda(\overline{G}) \geq 1$  for  $G[M] = P_4$  and get a contradiction among the former three cases. Without loss of generality, let



$G[M] = P_4 = u_1v_1u_2v_2$ . For any  $u_i, u_j \in V(G_1)$  ( $1 \leq i \leq n_1$ ),  $u_i$  and  $u_j$  are connected in  $\overline{G}$  since there exists a path  $u_iv_3u_j$  in  $\overline{G}$ ; for any  $v_i, v_j \in V(G_2)$  ( $1 \leq i \leq n_2$ ),  $v_i$  and  $v_j$  are connected in  $\overline{G}$  since there exists a path  $v_iu_3v_j$  in  $\overline{G}$ ; for any  $u_i \in V(G_1)$  and  $v_j \in V(G_2)$  ( $i \neq 3$  and  $j \neq 3$ ),  $u_i$  and  $u_j$  are connected in  $\overline{G}$  since there exists a path  $u_iv_3u_3v_j$  in  $\overline{G}$ . Since  $u_3v_j \in E(\overline{G})$  ( $1 \leq j \leq n_2$ ) and  $v_3u_i \in E(\overline{G})$  ( $1 \leq i \leq n_1$ ),  $\overline{G}$  is connected, a contradiction.

Now we consider the graph  $G$  such that  $G[M] = K_{1,n-3}$ . Assume  $n_1 \geq 2$ . If  $n_2 \geq 4$ , then we can check that  $\overline{G}$  is connected and get a contradiction. Therefore,  $n_2 = 3$ ,  $V(G_2) = \{v_1, v_2, v_3\}$  and  $V(G_1) = \{u_1, u_2, \dots, u_{n-3}\}$ . Since  $\lambda(G) = 3$ , it follows that  $v_1v_2, v_2v_3, v_1v_3 \in E(G)$ . Since  $v_iu_j \notin E(G)$  ( $1 \leq i \leq 3, 2 \leq j \leq n-3$ ), we have  $v_iu_j \in E(\overline{G})$ . If there exists some vertex  $u_j$  ( $2 \leq j \leq n-3$ ) such that  $u_1u_j \in E(\overline{G})$ , then  $\overline{G}$  is connected, a contradiction. So  $u_1u_j \in E(G)$  for  $2 \leq j \leq n-3$ . Thus  $d_G(u_1) = n-1$  (See Figure 3 (a)). From Lemma 5,  $\lambda_k(G) \geq 2$  for  $3 \leq k \leq n$  since  $\lambda(G) = 3$ , a contradiction.

Let us now assume  $n_1 = 1$ . Then  $V(G_1) = \{u_1\}$  and  $V(G_2) = \{v_1, v_2, \dots, v_{n-1}\}$ . If  $G[\{v_1, v_2, v_3\}] = 3K_1$  or  $G[\{v_1, v_2, v_3\}] = 2K_1 \cup K_2$ , then we have  $u_1v_j \in E(\overline{G})$  since  $u_1v_j \notin E(G)$  ( $4 \leq j \leq n-1$ ). From this together with the fact that  $\overline{G}$  is disconnected and  $v_1v_3, v_2v_3 \in E(\overline{G})$ ,  $v_iv_j \notin E(\overline{G})$  ( $1 \leq i \leq 3, 4 \leq j \leq n-1$ ), we have that  $v_iv_j \in E(G)$  ( $1 \leq i \leq 3, 4 \leq j \leq n-1$ ). Thus  $G$  contains a complete bipartite graph  $K_{3,n-3}$  as its subgraph (See Figure 3 (b) and (c)). From (1) of Lemma 1,  $\lambda_n(G) = \lfloor \frac{3(n-3)}{n-1} \rfloor \geq 2$  for  $n \geq 7$ , which implies  $\lambda_k(G) \geq 2$  for  $3 \leq k \leq n$  and  $n \geq 7$ . Since  $\lambda(G) = 3, n \geq 6$ . So we only need to consider the case  $n = 6$ . Thus  $G = H_i$  ( $1 \leq i \leq 4$ ) (See Figure 4). If  $G = H_i$  ( $2 \leq i \leq 4$ ), then  $\lambda_n(G) \geq 2$  for  $k = n = 6$  (See Figure 4 (b), (c), (d)). Therefore  $\lambda_k(G) \geq 2$  for  $3 \leq k \leq 6$ . If  $G = H_1$ , then  $\lambda_n(G) \leq \lfloor \frac{|E(G)|}{n-1} \rfloor = \lfloor \frac{9}{5} \rfloor = 1$  for  $k = n = 6$ . For  $k = 5$ , we can check that  $\lambda_3(G) \geq \lambda_4(G) \geq \lambda_5(G) \geq 2$  (See Figure 4 (e)). So  $G = K_{3,3}$  for  $k = n = 6$ .

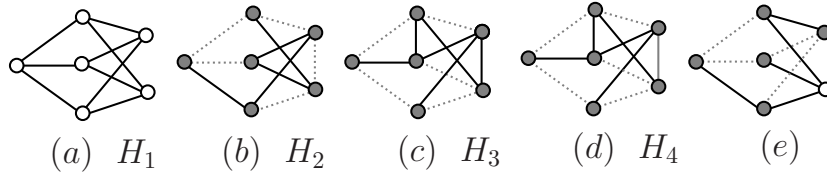


Figure 4. Graphs for Case 3 of Proposition 3.

Suppose  $G[\{v_1, v_2, v_3\}] = P_3$ . Without loss of generality, let  $v_1v_2, v_2v_3 \in E(G)$ . If  $3 \leq d_G(v_2) \leq n-2$  (See Figure 3 (d)), then there exists at least one vertex  $v_j$  such that  $v_2v_j \in E(\overline{G})$ , which results in  $v_1v_j, v_3v_j \notin E(\overline{G})$  ( $4 \leq j \leq n-1$ ) since  $u_1v_j \in E(\overline{G})$  ( $4 \leq j \leq n-1$ ),  $v_1v_3 \in E(\overline{G})$  and  $\overline{G}$  is disconnected. Thus  $v_1v_j, v_3v_j \in E(G)$  for each  $j$  ( $4 \leq j \leq n-1$ ). Since  $d(v_4) \geq \delta(G) \geq \lambda(G) = 3$ , we have  $v_4v_2 \in E(G)$  or there exists some vertex  $v_j$  ( $5 \leq j \leq n-1$ ) such that  $v_4v_j \in E(G)$ , which implies that  $G$  contains a subgraph  $K_{2,n-2}^{++}$  and so  $\lambda_n(G) \geq 2$  by (1) of Observation 3. From Proposition 2,  $\lambda_k(G) \geq 2$  for  $3 \leq k \leq n$ , a contradiction. If  $d_G(v_2) = n-1$  (See Figure 3 (e)), then  $\lambda_k(G) \geq 2$  for  $3 \leq k \leq n$  by Lemma 5 since  $\lambda(G) = 3$ , a contradiction.

Suppose  $G[\{v_1, v_2, v_3\}] = K_3$ . Without loss of generality, let  $v_1v_2, v_1v_3, v_2v_3 \in E(G)$ .

If  $d_G(v_1) = n - 1$  or  $d_G(v_2) = n - 1$  or  $d_G(v_3) = n - 1$  (See Figure 3 (f)), then by Lemma 5  $\lambda_k(G) \geq 2$  for  $3 \leq k \leq n$  since  $\lambda(G) = 3$ , a contradiction. If  $3 \leq d_G(v_i) \leq n - 2$  ( $1 \leq i \leq 3$ ), then  $\overline{G}$  is connected, a contradiction.  $\square$

We now investigate the upper bounds of  $\lambda_k(G) + \lambda_k(\overline{G})$  and  $\lambda_k(G) \cdot \lambda_k(\overline{G})$ .

**Lemma 6.** *Let  $G \in \mathcal{G}(n)$ . Then*

- (1)  $\lambda_k(G) + \lambda_k(\overline{G}) \leq n - \lceil k/2 \rceil$ ;
- (2)  $\lambda_k(G) \cdot \lambda_k(\overline{G}) \leq \lceil \frac{n - \lceil k/2 \rceil}{2} \rceil^2$ .

*Moreover, the two upper bounds are sharp.*

*Proof.* (1) Since  $G \cup \overline{G} = K_n$ ,  $\lambda_k(G) + \lambda_k(\overline{G}) \leq \lambda_k(K_n)$ . Combining this with Lemma 3,  $\lambda_k(G) + \lambda_k(\overline{G}) \leq n - \lceil \frac{k}{2} \rceil$ .

(2) The conclusion holds by (1).  $\square$

Let us focus on (1) of Lemma 6. If one of  $G$  and  $\overline{G}$  is disconnected, we can characterize the graphs attaining the upper bound by Lemma 4.

**Proposition 4.** *For any graph  $G$  of order  $n$ , if  $G$  is disconnected, then  $\lambda_k(G) + \lambda_k(\overline{G}) = n - \lceil \frac{k}{2} \rceil$  if and only if  $\overline{G} = K_n$  for  $k$  even;  $\overline{G} = K_n \setminus M$  for  $k$  odd, where  $M$  is an edge set such that  $0 \leq |M| \leq \frac{k-1}{2}$ .*

If both  $G$  and  $\overline{G}$  are all connected, we can obtain a structural property of the graphs attaining the upper bound although it seems too difficult to characterize them.

**Proposition 5.** *If  $\lambda_k(G) + \lambda_k(\overline{G}) = n - \lceil \frac{k}{2} \rceil$ , then  $\Delta(G) - \delta(G) \leq \lceil \frac{k}{2} \rceil - 1$ .*

*Proof.* Assume that  $\Delta(G) - \delta(G) \geq \lceil \frac{k}{2} \rceil$ . Since  $\lambda_k(\overline{G}) \leq \delta(\overline{G}) = n - 1 - \Delta(G)$ ,  $\lambda_k(G) + \lambda_k(\overline{G}) \leq \delta(G) + n - 1 - \Delta(G) \leq n - 1 - \lceil \frac{k}{2} \rceil$ , a contradiction.  $\square$

One can see that the graphs with  $\lambda_k(G) + \lambda_k(\overline{G}) = n - \lceil \frac{k}{2} \rceil$  must have a uniform degree distribution. Actually, we can construct a graph class to show that the two upper bounds of Lemma 6 are tight for  $k = n$ .

**Example 2.** Let  $n, r$  be two positive integers such that  $n = 4r + 1$ . From (1) of Lemma 1, we know that the *STP* number of the complete bipartite graph  $K_{2r, 2r+1}$  is  $\lfloor \frac{2r(2r+1)}{2r+(2r+1)-1} \rfloor = r$ , that is,  $\lambda_n(K_{2r, 2r+1}) = r$ . Let  $\mathcal{E}$  be the set of the edges of these  $r$  spanning trees in  $K_{2r, 2r+1}$ . Then there exist  $2r(2r+1) - 4r^2 = 2r$  remaining edges in  $K_{2r, 2r+1}$  except the edges in  $\mathcal{E}$ . Let  $M$  be the set of these  $2r$  edges. Set  $G = K_{2r, 2r+1} \setminus M$ . Then  $\lambda_n(G) = r$ ,  $M \subseteq E(\overline{G})$  and  $\overline{G}$  is a graph obtained from two cliques  $K_{2r}$  and  $K_{2r+1}$  by adding  $2r$  edges in  $M$  between them, that is, one endpoint of each edge belongs to  $K_{2r}$  and the other endpoint belongs to  $K_{2r+1}$ . Note that  $E(\overline{G}) = E(K_{2r}) \cup M \cup E(K_{2r+1})$ . Now we show that  $\lambda_n(\overline{G}) \geq r$ . As we know,  $K_{2r}$  contains  $r$  Hamiltonian paths, say  $P_1, P_2, \dots, P_r$ , and so does  $K_{2r+1}$ , say  $P'_1, P'_2, \dots, P'_r$ . Pick up  $r$  edges from  $M$ , say  $e_1, e_2, \dots, e_r$ , let  $T_i = P_i \cup P'_i \cup e_i$  ( $1 \leq i \leq r$ ). Then  $T_1, T_2, \dots, T_r$  are  $r$  spanning trees in  $\overline{G}$ , namely,  $\lambda_n(\overline{G}) \geq r$ . Since  $|E(\overline{G})| = \binom{2r}{2} + \binom{2r+1}{2} + 2r = 4r^2 + 2r$  and each spanning tree uses  $4r$

edges, these edges can form at most  $\lfloor \frac{4r^2+2r}{4r} \rfloor = r$  spanning trees, that is,  $\lambda_n(\overline{G}) \leq r$ . So  $\lambda_n(\overline{G}) = r$ .

Clearly,  $\lambda_n(G) + \lambda_n(\overline{G}) = 2r = \frac{n-1}{2} = n - \lceil \frac{n}{2} \rceil$  and  $\lambda_n(\overline{G}) \cdot \lambda_n(\overline{G}) = r^2 = \lfloor \frac{n-\lceil n/2 \rceil}{2} \rfloor^2$ , which implies that the upper bound of Lemma 6 is sharp.

Combining Lemmas 2 and 6, we give our main result.

**Theorem 2.** *Let  $G \in \mathcal{G}(n)$ . Then*

- (1)  $1 \leq \lambda_k(G) + \lambda_k(\overline{G}) \leq n - \lceil k/2 \rceil$ ;
- (2)  $0 \leq \lambda_k(G) \cdot \lambda_k(\overline{G}) \leq \lfloor \frac{n-\lceil k/2 \rceil}{2} \rfloor^2$ .

*Moreover, the upper and lower bounds are sharp.*

### 3 Nordhaus-Gaddum-type results in $\mathcal{G}(n, m)$

Achthan et. al. [1] restricted their attention to the subclass of  $\mathcal{G}(n, m)$  consisting of graphs with exactly  $m$  edges. They investigated the edge-connectivity, diameter and chromatic number parameters. For edge-connectivity  $\lambda(G)$ , they showed that  $\lambda(G) + \lambda(\overline{G}) \geq \max\{1, n-1-m\}$ . In this section, we consider a similar problem on the generalized edge-connectivity.

**Lemma 7.** *If  $M$  is an edge set of the complete graph  $K_n$  such that  $0 \leq m \leq \lfloor \frac{n}{3} \rfloor$  where  $|M| = m$ , then  $G = K_n \setminus M$  contains  $\ell$  edge-disjoint spanning trees, where  $\ell = \min\{n - 2m - 1, \lfloor \frac{n}{2} - \frac{2m}{n-1} \rfloor\}$ .*

*Proof.* Let  $\mathcal{P} = \bigcup_{i=1}^p V_i$  be a partition of  $V(G)$  with  $|V_i| = n_i$  ( $1 \leq i \leq p$ ), and  $\mathcal{E}_p$  be the set of edges between distinct parts of  $\mathcal{P}$  in  $G$ . It suffices to show that  $|\mathcal{E}_p| \geq \ell(|\mathcal{P}| - 1)$  so that we can use Nash-Williams-Tutte Theorem.

The case  $p = 1$  is trivial, thus we assume  $2 \leq p \leq n$ . Then  $|\mathcal{E}_p| \geq \binom{n}{2} - \sum_{i=1}^p \binom{n_i}{2} - |M| \geq \binom{n}{2} - \sum_{i=1}^p \binom{n_i}{2} - m$ . We will show that  $\binom{n}{2} - \sum_{i=1}^p \binom{n_i}{2} - m \geq \ell(p-1)$ , that is,  $\frac{n(n-1)}{2} - m - \ell(p-1) \geq \sum_{i=1}^p \binom{n_i}{2}$ . We only need to prove that  $\frac{n(n-1)}{2} - m - \ell(p-1) \geq \max\{\sum_{i=1}^p \binom{n_i}{2}\}$ . Since  $f(n_1, n_2, \dots, n_p) = \sum_{i=1}^p \binom{n_i}{2}$  achieves its maximum value when  $n_1 = n_2 = \dots = n_{p-1} = 1$  and  $n_p = n - p + 1$ , we need the inequality  $\frac{n(n-1)}{2} - m - \ell(p-1) \geq \binom{1}{2}(p-1) + \binom{n-p+1}{2}$ , that is,  $\frac{n(n-1)}{2} - m - \frac{(n-p+1)(n-p)}{2} \geq \ell(p-1)$ . Actually,  $\ell \leq \frac{n(n-1) - (n-p+1)(n-p) - 2m}{2(p-1)}$  is our required inequality, namely,  $\ell \leq n - \frac{1}{2} - (\frac{p-1}{2} + \frac{2m}{p-1})$ . Since  $f(x) = \frac{x}{2} + \frac{2m}{x}$  achieves its maximum value  $\max\{2m + \frac{1}{2}, \frac{n-1}{2} + \frac{2m}{n-1}\}$  when  $1 \leq x \leq n-1$ , we need  $\ell \leq \min\{n - 2m - 1, \frac{n}{2} - \frac{2m}{n-1}\}$ . Since this inequality holds for  $0 \leq m \leq \lfloor \frac{n}{3} \rfloor$ , we have  $|\mathcal{E}_p| \geq \binom{n}{2} - \sum_{i=1}^p \binom{n_i}{2} - |M| \geq \ell(p-1)$ . From Theorem 1, we know that  $G$  has  $\ell$  edge-disjoint spanning trees.  $\square$

**Lemma 8.** *Let  $G \in \mathcal{G}(n, m)$ . For  $n \geq 6$ , we have*

- (1)  $\lambda_k(G) + \lambda_k(\overline{G}) \geq L(n, m)$ , where

$$L(n, m) = \begin{cases} \max\{1, \lfloor \frac{1}{2}(n-2-m) \rfloor\} & \text{if } \lfloor \frac{n}{3} \rfloor + 1 \leq m \leq \binom{n}{2}, \\ \min\{n-2m-1, \lfloor \frac{n}{2} - \frac{2m}{n-1} \rfloor\} & \text{if } 0 \leq m \leq \lfloor \frac{n}{3} \rfloor. \end{cases}$$

$$(2) \lambda_k(G) \cdot \lambda_k(\overline{G}) \geq 0.$$

*Proof.* (1) Since at least one of  $G$  and  $\overline{G}$  must be connected, we have  $\lambda_k(G) + \lambda_k(\overline{G}) \geq 1$ . For  $m < n - 1$ ,  $\lambda_k(G) + \lambda_k(\overline{G}) \geq \lfloor \frac{1}{2}\lambda(G) \rfloor + \lfloor \frac{1}{2}\lambda(\overline{G}) \rfloor \geq \lfloor \frac{1}{2}(\lambda(G) + \lambda(\overline{G}) - 1) \rfloor \geq \lfloor \frac{1}{2}(\max\{1, n - 1 - m\} - 1) \rfloor \geq \lfloor \frac{1}{2}(n - 2 - m) \rfloor$  by Proposition 1. So  $\lambda_k(G) + \lambda_k(\overline{G}) \geq \max\{1, \lfloor \frac{1}{2}(n - 2 - m) \rfloor\}$ . In particular, for  $0 \leq m \leq \lfloor \frac{n}{3} \rfloor$ , we can give a better lower bound of  $\lambda_k(G) + \lambda_k(\overline{G})$  by Lemma 7, that is,  $\lambda_k(G) + \lambda_k(\overline{G}) = \lambda_k(\overline{G}) \geq \lambda_n(\overline{G}) \geq \min\{n - 2m - 1, \lfloor \frac{n}{2} - \frac{2m}{n-1} \rfloor\}$ .

To show the sharpness of the above lower bound for  $\lfloor \frac{n}{3} \rfloor + 1 \leq m \leq \binom{n}{2}$ , we consider the graph  $G = K_{1, n-2} \cup K_1$ . Then  $m = n - 2$  and  $\overline{G}$  is a graph obtained from a complete graph  $K_{n-1}$  by attaching a pendant edge. Clearly,  $\lambda_k(G) = 0$  and  $\lambda_k(\overline{G}) = 1$ . So  $\lambda_k(G) + \lambda_k(\overline{G}) = 1 = \max\{1, \lfloor \frac{1}{2}(n - 2 - m) \rfloor\}$ . To show the sharpness of the above lower bound for  $0 \leq m \leq \lfloor \frac{n}{3} \rfloor$ , we consider the graph  $G = nK_1$ . Thus  $m = 0$  and  $\overline{G} = K_n$ . Since  $\lambda_n(G) + \lambda_n(\overline{G}) = 0 + \lfloor \frac{n}{2} \rfloor = \min\{n - 2 \cdot 0 - 1, \lfloor \frac{n}{2} - \frac{2 \cdot 0}{n-1} \rfloor\}$ , that is, the lower bound is sharp for  $k = n$ .

(2) The inequality follows from Theorem 2.  $\square$

It was pointed out by Harary [9] that given the number of vertices and edges of a graph, the largest connectivity possible can also be read out of the inequality  $\kappa(G) \leq \lambda(G) \leq \delta(G)$ .

**Theorem 3.** [9] For each  $n, m$  with  $0 \leq n - 1 \leq m \leq \binom{n}{2}$ ,

$$\kappa(G) \leq \lambda(G) \leq \left\lfloor \frac{2m}{n} \right\rfloor,$$

where the maximum are taken over all graphs  $G \in \mathcal{G}(n, m)$ .

Now we will study a similar problem for the generalized edge-connectivity, which will be used in (2) of Lemma 9.

**Corollary 2.** For any graph  $G \in \mathcal{G}(n, m)$  and  $3 \leq k \leq n$ ,  $\lambda_k(G) = 0$  for  $m < n - 1$ ;  $\lambda_k(G) \leq \lfloor \frac{2m}{n} \rfloor$  for  $m \geq n - 1$ .

*Proof.* Let  $G \in \mathcal{G}(n, m)$ . When  $0 \leq m < n - 1$ ,  $G$  must be disconnected and hence  $\lambda_k(G) = 0$ . If  $m \geq n - 1$ ,  $\lambda_k(G) \leq \lambda(G) \leq \lfloor \frac{2m}{n} \rfloor$  by (1) of Observation 1 and Theorem 3.  $\square$

Although the above bound of  $\lambda_k(G)$  is the same as  $\lambda(G)$ , the graphs attaining the upper bound seems to be very rare. Actually, we can obtain some structural properties of these graphs.

**Proposition 6.** For any  $G \in \mathcal{G}(n, m)$  and  $3 \leq k \leq n$ , if  $\lambda_k(G) = \lfloor \frac{2m}{n} \rfloor$  for  $m \geq n - 1$ , then

- (1)  $\frac{2m}{n}$  is not an integer;
- (2)  $\delta(G) = \lfloor \frac{2m}{n} \rfloor$ ;
- (3) for  $u, v \in V(G)$  such that  $d_G(u) = d_G(v) = \lfloor \frac{2m}{n} \rfloor$ ,  $uv \notin E(G)$ .

*Proof.* One can check that the conclusion holds for the case  $m = n - 1$ . Assume  $m \geq n$ . We claim that  $\frac{2m}{n}$  is not an integer. Otherwise, let  $r = \frac{2m}{n}$  be an integer. We will show that  $\lambda_k(G) \leq r - 1 = \frac{2m}{n} - 1$  and get a contradiction. If  $G$  has at least one vertex  $v_i$  such that  $d(v_i) > r$ , then, since the average degree of  $G$  is exactly  $r$ , there must be a vertex  $v_j$  whose degree  $d(v_j) < r$ . From (1) of Observation 1, we have  $\lambda_k(G) \leq \delta(G) \leq d(v_j) < r$ , that is,  $\lambda_k(G) \leq r - 1$ . If, on the other hand,  $G$  is a regular graph, then by (3) of Observation 1,  $\lambda_k(G) \leq \delta(G) - 1 = r - 1$ . So (1) holds.

For a graph  $G$  such that  $\frac{2m}{n}$  is not an integer,  $\lfloor \frac{2m}{n} \rfloor = \lambda_k(G) \leq \delta(G) \leq \lfloor \frac{2m}{n} \rfloor$ , that is,  $\delta(G) = \lfloor \frac{2m}{n} \rfloor$ . So (2) holds.

For  $u, v \in V(G)$  such that  $d_G(u) = d_G(v) = \lfloor \frac{2m}{n} \rfloor$ , we claim that  $uv \notin E(G)$ . Otherwise,  $uv \in E(G)$ . Since  $d_G(u) = d_G(v) = \delta(G) = \lfloor \frac{2m}{n} \rfloor$ ,  $\lambda_k(G) \leq \delta(G) - 1 = \lfloor \frac{2m}{n} \rfloor - 1$  by (3) of Observation 1, a contradiction. So (3) holds.  $\square$

**Corollary 3.** *For any graph  $G$  of order  $n$  and size  $m$ , if  $\frac{2m}{n}$  is an integer, then  $\lambda_k(G) \leq \frac{2m}{n} - 1$ .*

**Lemma 9.** *Let  $G \in \mathcal{G}(n, m)$ . Then*

(1)  $\lambda_k(G) + \lambda_k(\overline{G}) \leq M(n, m)$ , where

$$M(n, m) = \begin{cases} n - \lceil \frac{k}{2} \rceil & \text{if } m \geq n - 1, \\ & \text{or } k \text{ is even and } m = 0, \\ & \text{or } k \text{ is odd and } 0 \leq m \leq \frac{k-1}{2}; \\ n - \lceil \frac{k}{2} \rceil - 1 & \text{if } k \text{ is even and } 1 \leq m < n - 1, \\ & \text{or } k \text{ is odd and } \frac{k+1}{2} \leq m < n - 1. \end{cases}$$

(2)  $\lambda_k(G) \cdot \lambda_k(\overline{G}) \leq N(n, m)$ , where

$$N(n, m) = \begin{cases} 0 & \text{if } 0 \leq m \leq n - 2, \\ (\frac{2m}{n} - 1)(n - 2 - \frac{2m}{n}) & \text{if } m \geq n - 1 \text{ and } 2m \equiv 0 \pmod{n}, \\ \lfloor \frac{2m}{n} \rfloor (n - 2 - \lfloor \frac{2m}{n} \rfloor) & \text{otherwise.} \end{cases}$$

Moreover, these upper bounds are sharp.

*Proof.* From Theorem 2, (1) holds for  $m \geq n - 1$ . We have given a graph class to show that the upper bound is sharp. From Proposition 4,  $\lambda_k(G) + \lambda_k(\overline{G}) = \lambda_k(\overline{G}) = n - \lceil \frac{k}{2} \rceil$  for  $k$  even and  $m = 0$ , or  $k$  odd and  $0 \leq m \leq \frac{k-1}{2}$ . So for  $k$  even and  $1 \leq m < n - 1$ , or  $k$  odd and  $\frac{k+1}{2} \leq m < n - 1$ ,  $\lambda_k(G) + \lambda_k(\overline{G}) \leq n - \lceil \frac{k}{2} \rceil - 1$ .

To prove the sharpness of the bound for  $k$  odd and  $\frac{k+1}{2} \leq m < n - 1$ , we consider the graph  $G = K_{1, \frac{k+1}{2}} \cup (n - \frac{k+3}{2})K_1$ . Now  $\overline{G}$  is a graph obtained from the complete graph  $K_n$  by deleting all the edges of a star  $K_{1, \frac{k+1}{2}}$ . On one hand, by Lemma 4,  $\lambda_k(\overline{G}) \leq n - \frac{k+1}{2} - 1$ . On the other hand, by Lemma 4, we have  $\lambda_k(\overline{G} + e) = n - \frac{k+1}{2}$  for any  $e \notin E(\overline{G})$ , which implies that  $\lambda_k(\overline{G}) \geq n - \frac{k+1}{2} - 1$  (Note that  $\lambda_k(H \setminus e) \geq \lambda_k(H) - 1$  for a connected graph  $H$ , where  $e \in E(H)$ ). So  $\lambda_k(G) + \lambda_k(\overline{G}) = \lambda_k(\overline{G}) = n - \frac{k+1}{2} - 1$ . By the same reason, for

$k$  even and  $1 \leq m < n-1$  one can check that the graph  $G = K_2 \cup (n-2)K_1$  satisfies that  $\lambda_k(G) + \lambda_k(\overline{G}) = \lambda_k(\overline{G}) \geq n - \frac{k}{2} - 1$ .

(2) First, if  $0 \leq m \leq n-2$ , then  $G \in \mathcal{G}(n, m)$  is disconnected. So  $\lambda_k(G) \cdot \lambda_k(\overline{G}) = 0$ . Next if  $\frac{2m}{n} = r$  is an integer, then  $\frac{2e(\overline{G})}{n} = n-1-r$  is also an integer. From Corollary 3, we have  $\lambda_k(G) \leq r-1$  and  $\lambda_k(\overline{G}) \leq n-2-r$ . So  $\lambda_k(G) \cdot \lambda_k(\overline{G}) \leq (r-1)(n-2-r) = (\frac{2m}{n}-1)(n-2-\frac{2m}{n})$ . Finally, if  $2m = nr + \ell$  where  $1 \leq \ell \leq n-1$ , then  $\Delta(G) \geq r+1$ . By (1) of Observation 1,  $\lambda_k(\overline{G}) \leq \delta(\overline{G}) = n-1-\Delta(G) \leq n-2-r$ . So  $\lambda_k(G) \cdot \lambda_k(\overline{G}) \leq r(n-2-r) = \lfloor \frac{2m}{n} \rfloor (n-2 - \lfloor \frac{2m}{n} \rfloor)$ .  $\square$

To show the sharpness of the upper bound for  $m \geq n-1$  and  $2m \equiv 0 \pmod{n}$ , we consider the following example.

**Example 3.** Let  $G$  be a cycle  $C_n = w_1 w_2 \cdots w_n w_1$  ( $n \geq 9$ ). Since  $\frac{2m}{n} = 2$  is an integer,  $\lambda_3(G) = \frac{2m}{n} - 1 = 1$ . It suffices to prove that  $\lambda_3(\overline{G}) = n-2 - \frac{2m}{n} = n-4$ .

Choose  $S = \{x, y, z\} \subseteq V(C_n) = V(G)$ . We will show that  $\lambda(S) \geq n-4$ . If  $d_{C_n}(x, y) = 1$  and  $d_{C_n}(y, z) = 1$ , without loss of generality, let  $N_{C_n}(x) = \{x_1, y\}$  and  $N_{C_n}(z) = \{y, z_2\}$ , then the trees  $T_i = xw_i \cup yw_i \cup zw_i$  together with  $T_1 = xz \cup zx_1 \cup x_1y$  form  $n-4$  edge-disjoint  $S$ -trees (See Figure 5 (a)), namely,  $\lambda(S) \geq n-4$ , where  $\{w_1, w_2, \dots, w_{n-5}\} = V(G) \setminus \{x, y, z, x_1, z_2\}$ .

If  $d_{C_n}(x, y) = 2$  and  $d_{C_n}(y, z) = 1$ , without loss of generality, let  $N_{C_n}(x) = \{x_1, y_1\}$  and  $N_{C_n}(z) = \{y_1, z\}$  and  $N_{C_n}(z) = \{y, z_2\}$ , then the trees  $T_i = xw_i \cup yw_i \cup zw_i$  together with  $T_1 = xy \cup xz$  and  $T_2 = z_2x \cup z_2y \cup z_2y_1 \cup y_1z$  form  $n-4$  edge-disjoint  $S$ -trees (See Figure 5 (b)), namely,  $\lambda(S) \geq n-4$ , where  $\{w_1, w_2, \dots, w_{n-6}\} = V(G) \setminus \{x, y, z, x_1, y_1, z_2\}$ .

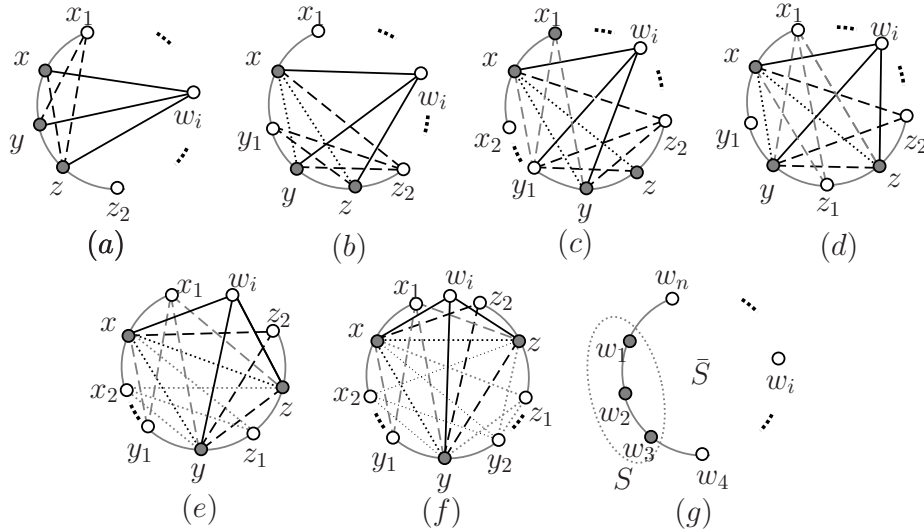


Figure 5. Graphs for Example 3.

If  $d_{C_n}(x, y) \geq 3$  and  $d_{C_n}(y, z) = 1$ , without loss of generality, let  $N_{C_n}(x) = \{x_1, x_2\}$  and  $N_{C_n}(z) = \{y_1, z\}$  and  $N_{C_n}(z) = \{y, z_2\}$ , then the trees  $T_i = xw_i \cup yw_i \cup zw_i$  together with  $T_1 = xy \cup xz$  and  $T_2 = z_2x \cup z_2y \cup z_2y_1 \cup y_1z$  and  $T_3 = xy_1 \cup y_1x_1 \cup x_1y \cup x_1z$  form  $n-4$  edge-disjoint  $S$ -trees (See Figure 5 (c)), namely,  $\lambda(S) \geq n-4$ , where  $\{w_1, w_2, \dots, w_{n-7}\} = V(G) \setminus \{x, y, z, x_1, x_2, y_1, z_2\}$ .



If  $d_{C_n}(x, y) = 2$  and  $d_{C_n}(y, z) = 2$ , without loss of generality, let  $N_{C_n}(x) = \{x_1, y_1\}$  and  $N_{C_n}(z) = \{y_1, z_1\}$  and  $N_{C_n}(z) = \{z_1, z_2\}$ , then the trees  $T_i = xw_i \cup yw_i \cup zw_i$  together with  $T_1 = xz \cup xy$  and  $T_2 = xz_2 \cup yz_2 \cup yz$  and  $T_3 = x_1y \cup x_1z \cup x_1z_1 \cup xz_1$  form  $n - 4$  edge-disjoint  $S$ -trees (See Figure 5 (d)), namely,  $\lambda(S) \geq n - 4$ , where  $\{w_1, w_2, \dots, w_{n-7}\} = V(G) \setminus \{x, y, z, x_1, y_1, z_1, z_2\}$ .

If  $d_{C_n}(x, y) \geq 3$  and  $d_{C_n}(y, z) = 2$ , without loss of generality, let  $N_{C_n}(x) = \{x_1, x_2\}$  and  $N_{C_n}(z) = \{y_1, z_1\}$  and  $N_{C_n}(z) = \{z_1, z_2\}$ , then the trees  $T_i = xw_i \cup yw_i \cup zw_i$  together with  $T_1 = xz \cup xy$  and  $T_2 = xz_2 \cup yz_2 \cup yz$  and  $T_3 = x_1y \cup x_1z \cup x_1y_1 \cup xy_1$  and  $T_4 = x_2y \cup x_2z \cup x_2z_1 \cup z_1x$  form  $n - 4$  edge-disjoint  $S$ -trees (See Figure 5 (e)), namely,  $\lambda(S) \geq n - 4$ , where  $\{w_1, w_2, \dots, w_{n-8}\} = V(G) \setminus \{x, y, z, x_1, x_2, y_1, y_2, z_2\}$ .

Suppose that  $d_{C_n}(x, y) \geq 3$  and  $d_{C_n}(y, z) \geq 3$ , without loss of generality, let  $N_{C_n}(x) = \{x_1, x_2\}$  and  $N_{C_n}(z) = \{y_1, y_2\}$  and  $N_{C_n}(z) = \{z_1, z_2\}$ . Then the trees  $T_i = xw_i \cup yw_i \cup zw_i$  together with  $T_1 = xz \cup xy$  and  $T_2 = xz_2 \cup yz_2 \cup yz$  and  $T_3 = xz_1 \cup yz_1 \cup y_2z_1 \cup y_2z$  and  $T_4 = x_1y \cup x_1z \cup x_1y_1 \cup y_1x$  and  $T_5 = x_2y \cup x_2z \cup x_2y_2 \cup y_2x$  form  $n - 4$  edge-disjoint  $S$ -trees (See Figure 5 (f)), namely,  $\lambda(S) \geq n - 4$ , where  $\{w_1, w_2, \dots, w_{n-9}\} = V(G) \setminus \{x, y, z, x_1, x_2, y_1, y_2, z_1, z_2\}$ .

From the arbitrariness of  $S$ , we know that  $\lambda_3(\overline{G}) \geq n - 4$  by definition. Now we show that  $\lambda_3(\overline{G}) \leq n - 4$  for  $\overline{G} = \overline{C_n}$ . Choose  $S = \{w_1, w_2, w_3\} \subseteq V(G) = V(C_n)$ . Then  $w_1w_n \in E(C_n)$  and  $w_3w_4 \in E(C_n)$ . Thus  $|E(\overline{G}[S])| = 1$  and  $|E_{\overline{G}}[S, \bar{S}]| = 3(n - 3) - 2$ , which implies that  $|E(\overline{G}[S]) \cup E_{\overline{G}}[S, \bar{S}]| = 3(n - 3) - 1$  (See Figure 5 (g)). One can see that each tree connecting  $S$  in  $\overline{G}$  uses at least 3 edges from  $E(\overline{G}[S]) \cup E_{\overline{G}}[S, \bar{S}]$ . Therefore  $\lambda_3(\overline{G}) \leq \frac{3(n-3)-1}{3} = n - 3 - \frac{1}{3}$ , which results in  $\lambda_3(\overline{G}) \leq n - 4$  since  $\lambda_3(\overline{G})$  is an integer. So  $\lambda_3(\overline{G}) = n - 4$  and  $\lambda_3(G) \cdot \lambda_3(\overline{G}) = \lambda_3(C_n) \cdot \lambda_3(\overline{C_n}) = 1 \cdot (n - 4) = (\frac{2m}{n} - 1)(n - 2 - \frac{2m}{n})$ . The upper bound is sharp.

For  $m \geq n - 1$  and  $\frac{2m}{n} = r + \ell(1 \leq \ell \leq n - 1)$ , let  $G = P_4$ . Then  $\lambda_3(G) = 1 = \lfloor \frac{6}{4} \rfloor = \lfloor \frac{2m}{n} \rfloor$  and  $\lambda_3(\overline{G}) = \lambda_3(P_4) = 1 = 4 - 2 - \lfloor \frac{6}{4} \rfloor = n - 2 - \lfloor \frac{2m}{n} \rfloor$ . So  $\lambda_3(G) \cdot \lambda_3(\overline{G}) = \lfloor \frac{2m}{n} \rfloor (n - 2 - \lfloor \frac{2m}{n} \rfloor)$ .

Combining with Lemmas 8 and 9, we can obtain the following result.

**Theorem 4.** *Let  $G \in \mathcal{G}(n, m)$ . For  $n \geq 6$ , we have*

- (1)  $L(n, m) \leq \lambda_k(G) + \lambda_k(\overline{G}) \leq M(n, m);$
- (2)  $0 \leq \lambda_k(G) \cdot \lambda_k(\overline{G}) \leq N(n, m),$

*where  $L(n, m), M(n, m), N(n, m)$  are defined in Lemmas 8 and 9.*

*Moreover, the upper and lower bounds are sharp.*

## References

- [1] N. Achuthan, N. R. Achuthan, L. Caccetta, *On the Nordhaus-Gaddum problems*, Australasian J. Combin. 2(1990), 5-27.
- [2] Y. Alavi, J. Mitchem, *The connectivity and edge-connectivity of complementary graphs*, Lecture Notes in Math. 186(1971), 1-3.
- [3] M. Aouchiche, P. Hansen, *A survey of Nordhaus-Gaddum type relations*, Discrete Appl. Math., 2012.
- [4] J. Bondy, U. Murty, *Graph Theory*, GTM 244, Springer, 2008.

- [5] G. Chartrand, S.F. Kappor, L. Lesniak, D.R. Lick, *Generalized connectivity in graphs*, Bull. Bombay Math. Colloq. 2(1984), 1-6.
- [6] G. Chartrand, F. Okamoto, P. Zhang, *Rainbow trees in graphs and generalized connectivity*, Networks 55(4)(2010), 360-367.
- [7] M. Grötschel, *The Steiner tree packing problem in VLSI design*, Math. Program. 78(1997), 265-281.
- [8] M. Grötschel, A. Martin, R. Weismantel, *Packing Steiner trees: A cutting plane algorithm and computational results*, Math. Program. 72(1996), 125-145.
- [9] F. Harary, *The maximum connectivity of a graph*, Proc. Nat. Acad. Sci. USA 1142-1146.
- [10] H. Li, X. Li, Y. Sun, *The generalied 3-connectivity of Cartesian product graphs*, Discrete Math. Theor. Comput. Sci. 14(1)(2012), 43-54.
- [11] S. Li, X. Li, Note on the hardness of generalized connectivity, J. Combin. Optim. 24(2012), 389-396.
- [12] S. Li, X. Li, W. Zhou, *Sharp bounds for the generalized connectivity  $\kappa_3(G)$* , Discrete Math. 310(2010), 2147-2163.
- [13] X. Li, Y. Mao, Y. Sun, *On the generalized (edge-)connectivity*, arXiv:1112.0127 [math.CO] 2011.
- [14] C.St.J.A, Nash-Williams, *Edge-disjonint spanning trees of finite graphs*, J. London Math. Soc. 36(1961), 445-450.
- [15] N. Sherwani, *Algorithms for VLSI physical design automation*, 3rd Edition, Kluwer Acad. Pub., London, 1999.
- [16] W. Tutte, *On the problem of decomposing a graph into  $n$  connected factors*, J. London Math. Soc. 36(1961), 221-230.
- [17] E. Palmer, *On the spanning tree packing number of a graph: a survey*, Discrete Math. 230(2001), 13-21.
- [18] M. Kriesell, *Edge-disjoint trees containing some given vertices in a graph*, J. Combin. Theory, Ser.B, 88(2003), 53-65.
- [19] M. Kriesell, *Edge-disjoint Steiner trees in graphs without large bridges*, J. Combin. Theory, Ser.B, 62(2009), 188-198.
- [20] H. Wu, D. West, *Packing Steiner trees and  $S$ -connectors in graphs*, J. Combin. Theory, Ser.B, 102(2012), 186-205.